

Separability Idempotents and Multiplier Algebras

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Abstract

Consider two non-degenerate algebras B and C over the complex numbers. We study a certain class of idempotent elements E in the multiplier algebra $M(B \otimes C)$, called *separability idempotents*. The conditions imply the existence of anti-isomorphisms $S : B \rightarrow C$ and $S' : C \rightarrow B$ satisfying

$$E(b \otimes 1) = E(1 \otimes S(b)) \quad \text{and} \quad (1 \otimes c)E = (S'(c) \otimes 1)E.$$

for all $b \in B$ and $c \in C$. There also exist what we call left and right integrals. A *left integral* is a linear functional φ on C such that $(\iota \otimes \varphi)E = 1$ in $M(B)$. Similarly a *right integral* is a linear functional ψ on B so that $(\psi \otimes \iota)E = 1$. Here we use ι for the identity map. Moreover the conditions on E are such that these formulas make sense and that these integrals are unique and faithful. The notion is somewhat more restrictive than what is generally considered in the case of (finite-dimensional) algebras with identity. But for various reasons, it seems to be quite natural to consider this type of separability idempotents in the case of non-degenerate algebras possibly without an identity.

One example is coming from a discrete quantum group (A, Δ) . Here we take $B = C = A$ and $E = \Delta(h)$ where h is the normalized cointegral. The anti-isomorphisms coincide with the original antipode and the same is true for the integrals. Another example is obtained from a regular weak multiplier Hopf algebra (A, Δ) . Now B and C are the images of the source and target maps. For E we take the canonical idempotent (that is $\Delta(1)$ if A has an identity). Again the anti-isomorphisms come from the antipode of the weak multiplier Hopf algebra. These two examples have motivated the study of separability idempotents as used in this paper.

We know that the integrals in both examples give rise to a duality. This duality also exists in the case of any separability idempotent as studied in this paper. We treat this extended notion of duality.

We pay special attention to the case of a self-adjoint separability idempotent when the $*$ -algebras B and C are operator algebras. This happens when the integrals are positive linear functionals. In fact, also this case has motivated the somewhat more restrictive notion of a separability idempotent as defined in this paper.

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0. Introduction

Consider the algebra $M_n(\mathbb{C})$ of $n \times n$ complex matrices and denote it by A . It is an algebra over \mathbb{C} with an identity. Let (e_{ij}) be a set of matrix units in A . Define $E \in A \otimes A$ by

$$E = \frac{1}{n} \sum_{i,j=1}^n e_{ij} \otimes e_{ij}.$$

Then E has the following properties.

First it is *an idempotent*, that is $E^2 = E$ in the algebra $A \otimes A$. If we consider A with its natural $*$ -algebra structure, then E is *self-adjoint*. Next, the legs of E are all of A in the following sense. The left leg of E is defined as the smallest subspace V of A so that $E \in V \otimes A$. Then we must have $V = A$. Similarly for the right leg. In $A \otimes A$ we have the equalities

$$E(1 \otimes A) = E(A \otimes 1) \quad \text{and} \quad (1 \otimes A)E = (A \otimes 1)E.$$

Consider the linear map $S : A \rightarrow A$ defined by $S(e_{ij}) = e_{ji}$. Then S is an anti-isomorphism of A and it satisfies

$$E(a \otimes 1) = E(1 \otimes S(a)) \quad \text{and} \quad (1 \otimes a)E = (S(a) \otimes 1)E$$

for all $a \in A$. Here S is involutive in the sense that $S^2 = \iota$ (where we use ι for the identity map on A). It also satisfies $S(a^*) = S(a)^*$ for all a when we consider A with its natural $*$ -algebra structure. We have that

$$m(S \otimes \iota)E = 1 \quad \text{and} \quad m(\iota \otimes S)E = 1$$

where again ι is the identity map and where m stands for the multiplication map on A , seen as a linear map from $A \otimes A$ to A . Finally, if τ is the trace on A , normalized so that $\tau(1) = n$, and if φ and ψ are defined as $n\tau$, we have

$$(\iota \otimes \varphi)E = 1 \quad \text{and} \quad (\psi \otimes \iota)E = 1.$$

This means that A is a *separable algebra* and that E is a *separability idempotent* in $A \otimes A$. Observe that this approach is slightly different from the more common treatments of separable algebras and separability idempotents as we find this e.g. in [Pe] and [W]. We will give more comments on these differences in the paper.

In this paper, we will consider the case of algebras that are *not necessarily finite-dimensional* and are also *not required to be unital*. We will consider separability idempotents in a sense that will be close to the above description. We will show how the known concept in the case of an algebra with identity relates with our approach to the more general case.

We have started to study this concept as a consequence of our work on weak multiplier Hopf algebras where it naturally pops up. This has influenced our approach. We will explain this and the relation with weak multiplier Hopf algebras further.

Separability idempotents

Our *starting point* is a pair of non-degenerate algebras B and C over the complex numbers and an idempotent element E in the multiplier algebra $M(B \otimes C)$ of the tensor product algebra $B \otimes C$, satisfying certain requirements as above. Some of the properties described in the simple case of $M_n(\mathbb{C})$ will become axioms while some others will be proven from the axioms. As it turns out however, not all of the properties will remain true and some need to be weakened.

The main differences are due to the fact that we do not assume the algebras to be finite-dimensional and unital. We will have the equivalent of the anti-isomorphism S above, but it will no longer be involutive and in the $*$ -algebra case, even though we will then assume that E is self-adjoint, S will no longer be a $*$ -map.

We refer to Section 1 of this paper for a precise definition of a *separability idempotent* and the general results.

When writing this paper, we have been *motivated by two special cases*.

First consider a *discrete quantum group*. By this we mean a multiplier Hopf algebra (A, Δ) with a (left) cointegral h so that $\varepsilon(h) = 1$ where ε is the counit of (A, Δ) . Then we consider $\Delta(h)$ as sitting inside the multiplier algebra $M(A \otimes A)$. This will be a separability idempotent in the sense of this paper and the antipode S will play the role of the anti-isomorphisms. It is known that S not necessarily satisfies $S^2 = \iota$ and that there are multiplier Hopf $*$ -algebras that are discrete quantum groups where S is not a $*$ -map. The trace τ in the case of $M_n(\mathbb{C})$ will be replaced by the left and the right integral on the discrete quantum group. We refer to Section 4 where this is explained in detail (and where also references are given). As we already have mentioned, this case has been a source of inspiration for the development of the general theory as we will show also further.

The second special case we will use to illustrate all this comes from a (regular) *weak multiplier Hopf algebra* (A, Δ) . In this case, the algebras B and C are the images $\varepsilon_s(A)$ and $\varepsilon_t(A)$ of the source and target maps ε_s and ε_t and E is the canonical idempotent playing the role of $\Delta(1)$ for this weak multiplier Hopf algebra. Again the antipode S gives the anti-isomorphisms. Once more, we refer to Section 4. In fact, it is this example that has led us to start with the investigation of separability idempotents as studied in this paper. We have used it in [VD-W4] to construct examples of weak multiplier Hopf algebras.

Content of the paper

In *Section 1* we first give the main definitions and the main results. The starting point is a pair of non-degenerate algebras B and C over the complex numbers and an idempotent E in the multiplier algebra $M(B \otimes C)$ with certain properties. As a consequence we find unique anti-isomorphisms $S : B \rightarrow C$ and $S' : C \rightarrow B$ satisfying

$$E(b \otimes 1) = E(1 \otimes S(b)) \quad \text{and} \quad (1 \otimes c)E = (S'(c) \otimes 1)E$$

for $b \in B$ and $c \in C$. These anti-isomorphisms play a crucial role in the further treatment. In this first section, we also discuss the relation with the known concepts in the literature and explain the differences. We will give more motivation here. Simple examples are already considered in this section.

In *Section 2*, we consider integrals on the algebras B and C . These are linear functionals ψ on B and φ on C so that $(\psi \otimes \iota)E = 1$ and $(\iota \otimes \varphi)E = 1$. These formulas will have a meaning in the multiplier algebras $M(C)$ and $M(B)$ respectively. The integrals always exist, are unique and faithful. There also exists modular automorphisms and these can be expressed in terms of the anti-isomorphisms S and S' .

Just as in the case of a discrete quantum group and a regular weak multiplier Hopf algebra, these integrals can be used to consider the duals \widehat{B} and \widehat{C} . They do not have the structure of an algebra (because there is no coproduct on B or C). However, there is some extra structure which nice properties. In this section, we only treat duality briefly. We consider more properties in the next section where we consider the involutive case. It turns out that in this involutive case, some of these results about duality seem to be more interesting.

So in *Section 3* we assume that the algebras B and C are $*$ -algebras and that the separability idempotent is a self-adjoint projection. We have the obvious relations of the anti-isomorphisms S and S' with the involution.

The interesting special case is the one where the integrals are positive linear functionals. It is shown that this is the case if and only if the underlying algebras B and C are operator algebras. In this case it turns out that they need to be direct sums of matrix algebras. More concrete formulas for this case are found in *Section 4*.

In *Section 3* we build further on the duality that we started to study in the previous section. As mentioned already, we get some nicer results, like a Plancherel type formula for the Fourier transforms.

Section 4 is devoted to special cases and examples. Here we consider the two examples as mentioned earlier. But before that, we consider some more basic examples. First we take for B and C the algebra M_n of $n \times n$ matrices over the complex numbers \mathbb{C} as in the beginning of the introduction. And we modify the earlier example in such a way that the anti-isomorphisms are no longer taking the transpose of matrices. We also discuss the construction with direct sums and we see that in some sense, these matrix examples are universal building blocks. This is correct in the case of involutive algebras with positive integrals. Then the underlying algebras are operator algebras and hence direct sums of matrix algebras in this setting.

Finally, in *Section 5* we draw some conclusions and discuss possible further research on this subject. One of the possible directions we explain is when actually the element E is no longer assumed to be an idempotent but instead satisfies $E^2 = 0$, while still sharing many of the other properties (in particular the existence of the anti-isomorphisms S and S').

Conventions and notations

We only work with algebras A over \mathbb{C} (although we believe that this is not essential and algebras over other fields can also be considered). We do not assume that they are unital

but we need that the product is non-degenerate. As it will turn out, all the algebras in our theory will have local units. Then of course, the product is automatically non-degenerate and also the algebra is idempotent.

When A is such an algebra, we use $M(A)$ for the multiplier algebra of A . When m is in $M(A)$, then by definition we can define am and mb in A for all $a, b \in A$ and we have $(am)b = a(mb)$. The algebra A sits in $M(A)$ as an essential two-sided ideal and $M(A)$ is the largest algebra having this property.

We consider $A \otimes A$, the tensor product of A with itself. It is again a non-degenerate algebra and we can consider the multiplier algebra $M(A \otimes A)$. The same is true for a multiple tensor product. We use σ for the flip map on $A \otimes A$, as well as for its natural extension to $M(A \otimes A)$.

When A is an algebra, we denote by A^{op} the algebra obtained from A by reversing the product. However, we will try to avoid this notion and only work with the given product in the original algebra A itself.

We use 1 for the identity in any of these multiplier algebras. On the other hand, we use ι for the identity map on A (or other spaces).

A linear functional f on A is called *faithful* if the bilinear map $A \times A \rightarrow \mathbb{C}$, mapping (a, b) to $f(ab)$ is non-degenerate. So, given $a \in A$, we have that $a = 0$ if either $f(ab) = 0$ for all b or $f(ba) = 0$ for all b . When we have a $*$ -algebra and when f is positive (i.e $f(a^*a) \geq 0$ for all a), then f is faithful if and only if $f(a^*a) = 0$ implies $a = 0$.

Some of our examples will involve matrix algebras. We will use M_n for the $(*)$ -algebra of all complex $n \times n$ matrices where $n = 1, 2, 3, \dots$

Finally, when V is any vector space, we will use V' for its linear dual space and evaluation of a functional $\omega \in V'$ in an element $v \in V$ will often be written as $\langle v, \omega \rangle$ or $\langle \omega, v \rangle$. So we will in particular not bother about the order in this case.

Basic references

For the theory of separability for algebras with identity, we refer to [Pe] and [W]. In this paper, we also refer often to our work on weak multiplier Hopf algebras. See [VD-W2] for an introduction to the subject (with a motivation of the axioms), [VD-W3] for the main theory and [VD-W4] for the results on the source and target maps and the source and target algebras. Especially this last paper is intimately related with this work on separability idempotents. Finally we are also using discrete quantum groups to illustrate some of our results. Here we refer to [VD-Z].

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1. Separability idempotents

In this section we start with the definition of a separability idempotent and we develop the theory. We illustrate a few aspects with simple examples. In particular we consider the well-known case of finite-dimensional algebras with identity. In Section 4 we will give more and less trivial examples. There we will also treat some special cases.

We start with two *non-degenerate algebras* B and C and an *idempotent element* E in the *multiplier algebra* $M(B \otimes C)$. Recall that the multiplier algebra $M(A)$ of a non-degenerate algebra is the largest algebra with identity that contains A as a dense two-sided ideal. If already A has an identity, then $M(A) = A$. If not, then $M(A)$ will be strictly larger. Given the two algebras B and C , we have natural imbeddings

$$B \otimes C \subseteq M(B) \otimes M(C) \subseteq M(B \otimes C).$$

If the algebras B and C have no identity, these inclusions in general are strict, also the last one.

The upcoming assumptions will imply that C is isomorphic with B^{op} , the algebra B but with the opposite product. And because the conditions almost fix E , we get a construction that will be dependent mostly only on the algebra B . The assumptions also imply certain restrictions on the type of algebras we can have. All this will be made precise in what follows in this section, as well as in the next sections.

First we consider the following assumptions.

1.1 Assumption We require that

$$(1.1) \quad E(1 \otimes c) \in B \otimes C \quad \text{and} \quad (b \otimes 1)E \in B \otimes C$$

for all $b \in B$ and $c \in C$. If we also have

$$(1.2) \quad (1 \otimes c)E \in B \otimes C \quad \text{and} \quad E(b \otimes 1) \in B \otimes C$$

for all $b \in B$ and $c \in C$, we call E *regular*. □

Here 1 is the identity in $M(B)$ or in $M(C)$. The reader should compare this with the notion of a (*regular*) *coproduct* on an algebra as defined e.g. in Definition 1.3 in [VD-W3]. Of course, these assumptions are void in the case of algebras with identity as then we have $E \in B \otimes C$. But in general, they are genuine assumptions, also the regularity assumption. To formulate the next assumption, we introduce the following definition (see also Definition 1.5 in [VD-W1]).

1.2 Definition By the *left leg* and the *right leg* of E we mean the smallest subspaces V of B and W of C respectively satisfying

$$E(1 \otimes c) \in V \otimes C \quad \text{and} \quad (b \otimes 1)E \in B \otimes W$$

for all $c \in C$ and $b \in B$. We call E *full* if the left and right legs of E are all of B and C respectively. \square

Again, the reader should compare this with the notion of a *full coproduct* as given in [VD-W3]. If E is regular, the left and right legs can also be defined with the factors $1 \otimes c$ and $b \otimes 1$ on the other side of E . This will give the same subspaces. See similar results in [VD-W1], [VD-W2] and [VD-W3] where related topics are treated.

We have the following easy consequences.

1.3 Proposition Let E be full. Then any element in B is a linear combination of elements of the form $(\iota \otimes \omega)(E(1 \otimes c))$ with $c \in C$ and ω a linear functional on C . Similarly, any element in C is a linear combination of elements of the form $(\omega \otimes \iota)((b \otimes 1)E)$ where $b \in B$ and with ω a linear functional on B . \square

In the regular case, we can also take expressions with b and c on the other side. The proof is easy, see e.g. Section 1 in [VD-W1] and in [VD-W2] where similar results are proven.

1.4 Proposition Let E be full. Then we have, for $b \in B$ and $c \in C$, that

$$\begin{aligned} (b \otimes 1)E = 0 & \quad \text{implies} \quad b = 0 \\ E(1 \otimes c) = 0 & \quad \text{implies} \quad c = 0. \end{aligned} \quad \square$$

In this case, the same results are true with the factors on the other side, even if E is not regular. This property is an easy consequence of the previous one, using non-degeneracy of the products in B and C .

Next, we formulate the most fundamental assumptions and we come to the main definition of this section.

1.5 Definition Assume that E is a regular and full idempotent in $M(B \otimes C)$. Then we call E a *separability idempotent* if also

$$(1.3) \quad E(B \otimes 1) = E(1 \otimes C) \quad \text{and} \quad (B \otimes 1)E = (1 \otimes C)E. \quad \square$$

It will be explained later why we use this terminology (see e.g. Proposition 1.10).

Before we continue, we would like to make the following remarks.

1.6 Remark i) In the case of unital algebras, some of the above conditions are automatically satisfied. This is the case for the conditions (1.1) and (1.2). In particular,

regularity is automatic. Of course, even for unital algebras, fullness is *not* automatic. The same is true for the conditions (1.3). However, it seems possible to prove the second condition from the first one in (1.3) and vice versa. See further in this section for an argument (cf. Examples 1.11).

ii) In the case of non-unital algebras, it is certainly also possible to find a set of weaker conditions that will imply the other ones. Moreover, one can expect that in the non-regular case, also weaker conditions can be formulated that still will make the upcoming construction possible.

iii) As we will see later, these conditions will imply that the algebras B and C are anti-isomorphic and that they are separable algebras. See e.g. [Pe] and [W]. \square

We will not investigate these questions further as we introduce this notion mainly with the purpose of constructing examples of weak multiplier Hopf algebras in [VD-W4]. We refer to Section 5 where we discuss some open problems and possible further research on these objects.

We will now collect several properties that we can conclude from the above conditions. They will be used in the next two sections and also illustrated in Section 4. And again, the results are also used in [VD-W4] for constructing examples of weak multiplier Hopf algebras.

1.7 Proposition Assume that E is a separability idempotent in $M(B \otimes C)$. Then there are anti-isomorphisms $S : B \rightarrow C$ and $S' : C \rightarrow B$ given by

$$E(b \otimes 1) = E(1 \otimes S(b)) \quad \text{and} \quad (1 \otimes c)E = (S'(c) \otimes 1)E$$

for all $b \in B$ and $c \in C$.

Proof: Take any element $b \in B$. By the first assumption in (1.3), there exist an element $c \in C$ so that $E(b \otimes 1) = E(1 \otimes c)$. Because E is full, this element c is uniquely defined (cf. Proposition 1.4). Hence we can define a linear map $S : B \rightarrow C$ so that $c = S(b)$, or in other words so that $E(b \otimes 1) = E(1 \otimes S(b))$ for all b . Again by fullness (Proposition 1.4), this map is injective. It is surjective because of the condition (1.3) again. Moreover S is an anti-isomorphism because for all $b, b' \in B$ we have

$$E(1 \otimes S(bb')) = E(bb' \otimes 1) = E(b' \otimes S(b)) = E(1 \otimes S(b')S(b)).$$

In a completely similar way, we have an anti-isomorphism $S' : C \rightarrow B$ given by $(S'(c) \otimes 1)E = (1 \otimes c)E$ for all $c \in C$. \square

Remark that also conversely, if the anti-isomorphisms S and S' are given and satisfy the defining formulas in the formulation of the above proposition, the conditions (1.3) in Definition 1.5 will follow. In fact, in practice, this is mostly how these conditions are proven (see e.g. the examples in Section 4).

In the next proposition, we consider the multiplication maps $m_C : C \otimes C \rightarrow C$ and $m_B : B \otimes B \rightarrow B$.

1.8 Proposition If E is a separability idempotent in $M(B \otimes C)$, then

$$m_C(S \otimes \iota)(E(1 \otimes c)) = c \quad \text{and} \quad m_B(\iota \otimes S')((b \otimes 1)E) = b$$

for all $c \in C$ and all $b \in B$.

Proof: Take any element $c \in C$ and write c' for $m_C(S \otimes \iota)(E(1 \otimes c))$. Then we have

$$E(1 \otimes c') = E(E(1 \otimes c)) = E(1 \otimes c)$$

because $E^2 = E$. This implies $c' = c$ and it proves the first formula. Similarly for the second formula. \square

We think of the formulas above as

$$S(E_{(1)})E_{(2)} = 1 \quad \text{and} \quad E_{(1)}S'(E_{(2)}) = 1$$

with the Sweedler type notation $E = E_{(1)} \otimes E_{(2)}$. The interpretation is straightforward in the case of unital algebras. In the non-unital case, one still can interpret these formulas in the multiplier algebras $M(C)$ and $M(B)$ (as we have done in the formulation of the results).

We would like to add one more comment here. Up to Proposition 1.8, we did not use yet that E is idempotent. In fact, without this assumption, we can define a central element e in $M(C)$ satisfying $m_C(S \otimes \iota)(E(1 \otimes c)) = ec$ for all $c \in C$. Indeed, when $e = m_C(S \otimes \iota)(E)$, we find for all $b \in B$ that

$$\begin{aligned} eS(b) &= m_C(S \otimes \iota)(E(1 \otimes S(b))) \\ &= m_C(S \otimes \iota)(E(b \otimes 1)) \\ &= S(b)m_C(S \otimes \iota)(E) = S(b)e \end{aligned}$$

proving that e is a central element in $M(C)$. We see that it will be 1 if and only if $E^2 = E$. If we would have $E^2 = 0$, it would follow that $e = 0$. We consider this possibility in Section 5.

1.9 Proposition If there is a separability idempotent, then the algebras B and C will have local units. In particular, they are idempotent (i.e. $B^2 = B$ and $C^2 = C$).

Proof: We will show that $b \in bB$ and $b \in Bb$ for all $b \in B$. We know from [Ve] that this implies that B has local units.

Take any element $b \in B$ and assume that ω is a linear functional on B so that ω is 0 on bB . We claim that then $\omega(b) = 0$ and this will imply that $b \in bB$.

Indeed, by assumption we have

$$(\omega \otimes \iota)((b \otimes 1)E(b' \otimes 1)) = 0$$

for all $b' \in B$. Write $(b \otimes 1)E$ as $\sum_i b_i \otimes c_i$ with $b_i \in B$ and $c_i \in C$. We can assume that the $\{c_i\}$ are linearly independent. Then it follows that $\omega(b_i b') = 0$ for all i

and all b' . Now, replace b' by $S'(c_i)$ where S' is the anti-isomorphism from C to B , obtained in Proposition 1.7, and take the sum over i . By Proposition 1.8, we will have $\sum_i b_i S'(c_i) = b$. This shows that $\omega(b) = 0$.

Next assume that $b \in B$ and that ω is a linear functional on B so that ω is 0 on Bb . Then it follows that

$$(\omega \otimes \iota)((b' \otimes 1)E(b \otimes 1)) = 0$$

for all $b' \in B$. We write again $E(b \otimes 1)$ as $\sum_i b_i \otimes c_i$ with $b_i \in B$ and $c_i \in C$ and the $\{c_i\}$ linearly independent. As before it follows that $\omega(b'b_i) = 0$ for all i and all b' . Now replace b' by $S^{-1}(c_i)$ where S is the anti-isomorphism from B to C , obtained in Proposition 1.7, and take the sum over i . Now we use the first formula in Proposition 1.8 and again we find $\omega(b) = 0$. So we find that also $b \in Bb$ for all b .

A similar argument will show that also C has local units. On the other hand, because C is anti-isomorphic with B , also this implies that C will have local units. This completes the proof. \square

We can say more about the structure of the algebras. Indeed, because e.g. $(b \otimes 1)E \in B \otimes C$, it follows from the fullness of E that bB is finite-dimensional for all b . Similarly Bb is finite-dimensional for all b . Then BbB is a finite-dimensional two-sided ideal of B for any element b . In particular, any element of B sits in a finite-dimensional two-sided ideal. In Section 3, where we consider the involutive case, we will see that this will imply that the algebras are direct sums of matrix algebras.

The following result should justify the use of the term 'separability idempotent'.

1.10 Proposition Assume as before that E is a separability idempotent in $M(B \otimes C)$. Consider $B \otimes C$ as a right $(B \otimes C)$ -module with multiplication as the action. Also consider C as a right module over $B \otimes C$ with action $x \blacktriangleleft (b \otimes c) = S(b)xc$ where S is the anti-isomorphism from B to C as given in Proposition 1.7. Define a linear map $m : B \otimes C \rightarrow C$ by $m(b \otimes c) = S(b)c$. Then m is a module map and the short exact sequence

$$0 \longrightarrow \text{Ker}(m) \longrightarrow B \otimes C \xrightarrow{m} C \longrightarrow 0$$

is split.

Proof: It is straightforward to show that m is a module map for the given right module structures on $B \otimes C$ and C . If we define $\gamma : C \rightarrow B \otimes C$ by $\gamma(c) = E(1 \otimes c)$, it follows from the equation $E(b \otimes 1) = E(1 \otimes S(b))$ for $b \in B$ that γ is a module map. And because $m_C(S \otimes \iota)(E(1 \otimes c)) = c$, we see that $m\gamma(c) = c$ for all $c \in C$. This proves the result. \square

Remark that we are only using 'half' of the assumptions. We just need the anti-isomorphism $S : B \rightarrow C$ and not the other one $S' : C \rightarrow B$. Using the anti-isomorphism S' , we can formulate a similar result. This indicates that we have a condition that is more restrictive than what is common in the literature.

We will see this more clearly when we consider our notion in the finite-dimensional unital case. This seems to be the right place to do so. It also illustrates some of our conditions.

1.11 Examples i) Let A be a finite-dimensional algebra with identity. Assume that $E = \sum_i u_i \otimes v_i$ is an element in $A \otimes A$ satisfying

$$(1.4) \quad \sum_i u_i v_i = 1$$

and that

$$(1.5) \quad \sum_i a u_i \otimes v_i = \sum_i u_i \otimes v_i a$$

for all $a \in A$. Consider two copies of E , written as

$$E = \sum_i u_i \otimes v_i \quad \text{and} \quad E = \sum_j u'_j \otimes v'_j.$$

Let $B = A$ and $C = A^{\text{op}}$. Then the product E^2 in $B \otimes C$, written in $A \otimes A$ is

$$\sum_{i,j} u'_j u_i \otimes v_i v'_j = \sum_{i,j} u_i \otimes v_i u'_j v'_j = \sum_i u_i \otimes v_i$$

where we have used (1.5) with one copy of E and u'_j in the place of a and formula (1.4) for the other copy. This shows that E is idempotent in $B \otimes C$. Moreover, in $B \otimes C$ equation (1.5) is written as

$$(S'(c) \otimes 1)E = (1 \otimes c)E$$

if we define the linear map $S' : C \rightarrow B$ as the identity map from A^{op} to A . It is clearly an anti-isomorphism. We see that we get part of the requirements to have a separability idempotent in the sense of our Definition 1.5.

ii) Now assume that we take again the general situation of two non-degenerate algebras B and C and that we have a regular idempotent E in $M(B \otimes C)$ as defined in Assumption 1.1. Assume also that it is full in the sense of Definition 1.2. We then have the properties obtained in the Propositions 1.3 and 1.4. Next assume that we only have one of the assumptions in Definition 1.5, say that we only assume that

$$(B \otimes 1)E = (1 \otimes C)E.$$

Then we see as in Proposition 1.7 that there is an anti-isomorphism $S' : C \rightarrow B$ satisfying

$$(1.6) \quad (1 \otimes c)E = (S'(c) \otimes 1)E$$

for all c in C . We claim that the other assumption of (1.3) in Definition 1.5 automatically follows. Indeed, write two copies of E using the Sweedler type notation as

$$E = E_{(1)} \otimes E_{(2)} \quad \text{and} \quad E = E'_{(1)} \otimes E'_{(2)}.$$

Then we have, using the standard leg-numbering notation

$$\begin{aligned} E_{13}(1 \otimes E) &= E_{(1)} \otimes E'_{(1)} \otimes E_{(2)} E'_{(2)} \\ &= E_{(1)} \otimes S'(E_{(2)}) E'_{(1)} \otimes E'_{(2)} \\ &= E_{(1)} \otimes S'(S'^{-1}(E'_{(1)}) E_{(2)}) \otimes E'_{(2)} \\ &= E'_{(1)} E_{(1)} \otimes S'(E_{(2)}) \otimes E'_{(2)} \end{aligned}$$

where we have used (1.6) two times, first for one of the elements E and then for the other one. Of course, to make the argument precise, we need to multiply with elements of A at the right places (using regularity of E). If now we apply a linear functional ω on B on the middle factor in the above result, we find that

$$E(1 \otimes c) = E(b \otimes 1)$$

where

$$c = (\omega \otimes \iota)E \quad \text{and} \quad b = (\iota \otimes \omega \circ S')E.$$

Again, we have to consider a reduced functional to get this correct. This proves the claim. In particular, it follows that we have a regular separability idempotent E in the sense of Definition 1.5.

Compare this result with the formula $(S \otimes S')E = \sigma E$ that we will obtain in Proposition 1.13 below. It indeed follows from this formula that S' is given when S is given and vice versa, provided the legs of E are everything.

iii) If we now apply the result in ii) to the case considered in i), we see that E will be a separability idempotent in the sense of our Definition 1.5 provided the two legs of E are all of A . \square

The above discussion indicates several things.

First we see that our notion differs from what is commonly used in the literature. We require E to be full (i.e. the legs are the whole algebra). This is not a common condition. Consider e.g. the example in the introduction. But instead, take

$$E = \sum_i e_{i1} \otimes e_{i1}$$

where we use again matrix units (e_{ij}) in $M_n(\mathbb{C})$. This element will be a separability idempotent as in item i) above. But the legs will not be all of the algebra and so, it will not be a separability idempotent in the sense of this paper.

There is a notion of symmetric separability idempotents, but then this is more restrictive than ours (as it would correspond to the case where $S = S'$). The element E in the introduction is symmetric, but the derived separability idempotents we construct from it in Section 4 are not symmetric.

On the other hand, our more restrictive notion seems very natural from various points of view. Instead of considering only one side, we consider the two sides together. This is very natural in the $*$ -algebra case. Moreover, it pops up in the theories of discrete quantum groups and weak multiplier Hopf algebras as we will see in Section 4 where we consider more examples.

Finally, the case of a discrete quantum group seems to suggest another generalization, namely where the condition $E^2 = E$ is only one possibility and where also the case $E^2 = 0$ would be allowed. Again, we refer to Section 5.

Now we will see to what extent the separability idempotent is determined by the algebras. The following result is relatively simple to prove.

1.12 Proposition Let E and F be two separability idempotents. If the associated anti-isomorphisms S, S' are the same for E and F , then $E = F$.

Proof: On the one hand we have

$$EF = E(F_{(1)} \otimes F_{(2)}) = E(1 \otimes S(F_{(1)})F_{(2)}) = E$$

where we have used the Sweedler notation $F = F_{(1)} \otimes F_{(2)}$ and the fact that the anti-isomorphism S is the same for E and for F . On the other hand, we also have

$$EF = (E_{(1)} \otimes E_{(2)})F = ((E_{(1)}S'(E_{(2)}) \otimes 1)F = F$$

where again we have used the notation $E = E_{(1)} \otimes E_{(2)}$ and now the fact that the anti-isomorphism S' is the same for E and for F . Then $E = F$. \square

As an easy consequence of this result we get the following. We use the extension of the anti-isomorphisms to the multiplier algebras.

1.13 Proposition When E is a separability idempotent, then $(S \otimes S')E = \sigma E$ where S and S' are the associated anti-isomorphisms and where σ is the flip from $B \otimes C$ to $C \otimes B$, extended to the multiplier algebra.

Proof: Let $F = \sigma(S \otimes S')E$. Then it is clear that F will be a regular and full idempotent in $M(B \otimes C)$. It will be enough to show that it satisfies the assumptions of Definition 1.5 and that the associated anti-isomorphisms are again S and S' . When $b \in B$ we have

$$\begin{aligned} F(b \otimes 1) &= \sigma((S \otimes S')(E)(1 \otimes b)) = \sigma(S \otimes S')((1 \otimes S'^{-1}(b))E) \\ &= \sigma(S \otimes S')((b \otimes 1)E) = \sigma((S \otimes S')(E)(S(b) \otimes 1)) = F(1 \otimes S(b)). \end{aligned}$$

A similar argument will prove that also the second condition (1.3) in the Definition 1.5 is satisfied and that the associated anti-isomorphism is again S' . Then we can apply the previous result. \square

If we have two separability idempotents E_1 and E_2 with associated anti-isomorphisms S_1, S'_1 and S_2, S'_2 that are not the same, we have to look at the relation between the automorphisms $S'_1 S_1$ and $S'_2 S_2$ of B . We get the following result.

1.14 Proposition With the notations as above, we have that there exists automorphisms α_B and α_C of B and C respectively so that $E_2 = (\alpha_B \otimes \alpha_C)E_1$ if and only if the automorphisms $S'_1 S_1$ and $S'_2 S_2$ of B are conjugate.

Proof: First assume that there exist automorphisms α_B and α_C of B and C respectively so that $E_2 = (\alpha_B \otimes \alpha_C)E_1$. If we apply $\alpha_B \otimes \alpha_C$ on the equality $E_1(b \otimes 1) = E_1(1 \otimes S_1(b))$, we will find that $S_2(\alpha_B(b)) = \alpha_C(S_1(b))$ for all $b \in B$. Similarly, we get $S'_2(\alpha_C(c)) = \alpha_B(S'_1(c))$ for all $c \in C$. Combining the two results we find

$$S'_2 S_2 \circ \alpha_B = \alpha_B \circ S'_1 S_1 \quad \text{and} \quad S_2 S'_2 \circ \alpha_C = \alpha_C \circ S_1 S'_1.$$

This proves one direction. To prove the other direction, assume that we have given two separability idempotents E_1 and E_2 with associated anti-isomorphisms and that there exists an automorphism α_B of B so that $S'_2 S_2 \circ \alpha_B = \alpha_B \circ S'_1 S_1$. We now define the automorphism α_C of C by the relation $S'_2(\alpha_C(c)) = \alpha_B(S'_1(c))$. It follows that we get the other relation $S_2(\alpha_B(b)) = \alpha_C(S_1(b))$. Then we see that the two idempotents E_2 and $(\alpha_B \otimes \alpha_C)E_1$ have the same associated anti-isomorphisms. By Proposition 1.11, they have to be the same. This completes the proof. \square

For the automorphism $S'S$ of B , one can show that it is implemented by an invertible element in $M(B)$ if there is a faithful trace on C . Then there is also a faithful trace on B and SS' will be implemented on C by an invertible element in $M(C)$. We will prove this in the next section as it is related with the study of integrals.

2. Integrals and duality

As before let B and C be non-degenerate algebras and E a separability idempotent in $M(B \otimes C)$. In this section we will prove the *existence of integrals* as announced in the introduction. We will show that they are unique and faithful and that there exist modular automorphisms, just as in the case of integrals on regular (weak) multiplier Hopf algebras. We will also use these integrals to construct the reduced dual spaces \widehat{B} of B and \widehat{C} of C . Not much can be shown in this general case, but we will see in the next section, where we consider $*$ -algebras, that some interesting features can be proven for these duals.

Much of this section is inspired by the special cases that we study in the Section 4.

Integrals

The starting point and the main property of this section is the result of Proposition 2.1 below.

It follows from the assumptions (1.1) and (1.2) in the first section that for any linear functional ω on C , the element $(\iota \otimes \omega)E$ is defined in $M(B)$. Similarly $(\omega \otimes \iota)E$ is defined in $M(C)$ for any linear functional ω on B . This observation is important for the formulation of the following proposition.

2.1 Proposition There exists unique linear functionals ψ on B and φ on C satisfying

$$(\psi \otimes \iota)E = 1 \quad \text{and} \quad (\iota \otimes \varphi)E = 1$$

in $M(C)$ and $M(B)$ respectively. These functionals are faithful.

Proof: If φ is a linear functional on C so that $(\iota \otimes \varphi)E = 1$, then we will have $\varphi(c) = \sum_i \omega_i(b_i)$ if $c = \sum_i (\omega_i(b_i \cdot) \otimes \iota)E$ where $b_i \in B$ and ω_i is a linear functional on B for all i . Because any element $c \in C$ can be written as such a sum (cf. Proposition 1.3), this proves uniqueness of φ if it exists. However, it also suggest how to define φ and if we use the above formula, we see that φ will satisfy

$$(2.1) \quad (\iota \otimes \varphi)((b \otimes 1)E) = b$$

for all $b \in B$.

Now let us assume that we have a sum like $\sum_i (\omega_i(b_i \cdot) \otimes \iota)E$ and that this is equal to 0. Because E is full, this will imply that $\sum_i \omega_i(b_i b) = 0$ for all $b \in B$. Since we know that there are local units in B (cf. Proposition 1.9), it follows that also $\sum_i \omega_i(b_i) = 0$. Then we can complete the proof of the existence of φ satisfying the formula (2.1). Similarly we can treat the linear functional ψ .

We now prove that φ is faithful. Assume first that $c \in C$ and that $\varphi(cx) = 0$ for all $x \in C$. This implies that $(\iota \otimes \varphi)((1 \otimes c)E) = 0$. Then also $S'(c) = (\iota \otimes \varphi)((S'(c) \otimes 1)E) = 0$. Then $c = 0$. A similar argument will show that $c = 0$ if $\varphi(xc) = 0$ for all $x \in C$. Hence φ is a faithful functional on C . In a similar way one can show that ψ is faithful. \square

The following terminology is suggested by the example (see Proposition 4.7 Section 4).

2.2 Definition We call φ the *left integral* and ψ the *right integral*. \square

Clearly, from the equality $(S \otimes S')E = \sigma E$ proven in Proposition 1.13, we find that $\psi \circ S' = \varphi$ and $\varphi \circ S = \psi$. This follows from the uniqueness of these integrals. As a consequence, we find that φ is invariant for the automorphism SS' of C and that ψ is invariant under the automorphism $S'S$ of B . In particular we see that there is no non-trivial 'scaling constant' contrary to what we can have in the case of algebraic quantum groups.

We now prove that these integrals are *weak KMS-functionals*, just as integrals on algebraic quantum groups. By this we mean that there exist *modular automorphisms* for these functionals (cf. [VD2]).

2.3 Proposition Let φ and ψ be the left and right integrals, obtained in the previous proposition. Then there exist automorphisms σ of C and σ' of B so that

$$\varphi(cc') = \varphi(c'\sigma(c)) \quad \text{and} \quad \psi(bb') = \psi(b'\sigma'(b))$$

for all $b, b' \in B$ and $c, c' \in C$. In fact we have $\sigma(c) = S(S'(c))$ for $c \in C$ and $\sigma'(b) = S^{-1}(S'^{-1}(b))$ for $b \in B$.

Proof: We have

$$\begin{aligned} (\iota \otimes \varphi)((1 \otimes c)E) &= (\iota \otimes \varphi)((S'(c) \otimes 1)E) = S'(c) \\ &= (\iota \otimes \varphi)(E(S'(c) \otimes 1)) \\ &= (\iota \otimes \varphi)(E(1 \otimes S(S'(c)))) \end{aligned}$$

for all $c \in C$. This proves $\varphi(cc') = \varphi(c'\sigma(c))$ when $\sigma(c) = S(S'(c))$ for $c, c' \in C$. Similarly for ψ on B . \square

We know that the functionals are invariant under their modular automorphisms. In this case, the invariance of φ under σ is nothing else but the property that φ is invariant under SS' , a result that we obtained before the previous one. Similarly, the invariance of ψ under σ' is the same as the invariance of ψ under $S'S$ that we had before.

Before we continue with looking at the dual spaces, let us first relate the above property with the case where traces exist and where the automorphisms σ and σ' are implemented.

2.4 Proposition Let τ be a trace on B and let $q = (\tau \otimes \iota)E$. Then $q \in M(C)$, we have $cq = q\sigma(c)$ for all $c \in C$ and $\tau = \psi(S'(q) \cdot)$. Conversely, if $q \in M(C)$ and $cq = q\sigma(c)$ for all $c \in C$, then $\psi(S'(q) \cdot)$ is a trace on B . This trace is faithful if and only if q is invertible.

Proof: First suppose that τ is a trace on B and let $q = (\tau \otimes \iota)E$. We know that q is well-defined in $M(C)$.

For any $c \in C$ we have

$$\begin{aligned} cq &= (\tau \otimes \iota)((1 \otimes c)E) = (\tau \otimes \iota)((S'(c) \otimes 1)E) \\ &= (\tau \otimes \iota)(E(S'(c) \otimes 1)) = (\tau \otimes \iota)(E(1 \otimes S(S'(c)))) = qS(S'(c)). \end{aligned}$$

and this proves that $cq = q\sigma(c)$ for all $c \in C$.

Furthermore, because

$$q = (\psi \otimes \iota)((1 \otimes q)E) = (\psi \otimes \iota)((S'(q) \otimes 1)E)$$

we see that $\tau = \psi(S'(q) \cdot)$ when q is $((\tau \otimes \iota)E)$.

Conversely assume that $q \in M(C)$ and that $cq = q\sigma(c)$ for all $c \in C$. Define τ on B by $\tau(b) = \psi(S'(q)b)$. Then we have

$$(\tau \otimes \iota)E = (\psi \otimes \iota)(S'(q) \otimes 1)E = (\psi \otimes \iota)((1 \otimes q)E) = q.$$

The same sequence of equalities as earlier in this proof will now imply that τ is a trace.

It is clear from the relation $\tau = \psi(S'(q) \cdot)$ that τ will be faithful if q is invertible because we know that already ψ is faithful.

Conversely assume that τ is faithful. Then we can show that $qC = C$ and $Cq = C$. Indeed, assume e.g. that ω is a linear functional on C so that $\omega(cq) = 0$ for all $c \in C$. Then

$$(\tau \otimes \omega)((S'(c) \otimes 1)E) = (\tau \otimes \omega)((1 \otimes c)E) = \omega(cq) = 0$$

for all c and because τ is assumed to be faithful, we must have $(\iota \otimes \omega)E = 0$ in $M(B)$. Since E is assumed to be full, we must have $\omega = 0$. This proves that $Cq = C$. In a similar way we can show that $qC = C$.

Now take $c \in C$ and assume that $cq = 0$. This means that

$$(\tau \otimes \iota)((S'(c) \otimes 1)E) = (\tau \otimes \iota)((1 \otimes c)E) = cq = 0.$$

Again because E is full, this implies that $\tau(S'(c)b) = 0$ for all $b \in B$. Because τ is faithful, we get $c = 0$. Similarly we find $c = 0$ if $qc = 0$.

Because of these two properties, we can define a multiplier p by the formulas

$$p(qc) = c \quad \text{and} \quad (cq)p = c$$

for $c \in B$. And then clearly $pq = qp = 1$ in $M(C)$. Hence q is invertible in $M(C)$. This completes the proof. \square

Remark that at the end of the proof above, we find a necessary and sufficient condition for an element in $M(C)$ to be invertible. It holds for any non-degenerate algebra C .

The dual spaces \widehat{B} and \widehat{C}

We introduce the following subspaces of the dual spaces of B and C .

2.5 Definition We define

$$\widehat{B} = \{\psi(b \cdot) \mid b \in B\} \quad \text{and} \quad \widehat{C} = \{\varphi(\cdot c) \mid c \in C\}. \quad \square$$

Remark that we get the same space \widehat{B} if we put the elements b on the other side. Similarly for \widehat{C} .

Let us also introduce the following terminology and notations.

2.6 Notation For $b \in B$ and $c \in C$ we denote

$$\widehat{b} = \psi(b \cdot) \quad \text{and} \quad \widehat{c} = \varphi(\cdot c)$$

and we call \widehat{b} and \widehat{c} the Fourier transforms of b and c respectively. \square

Sometimes, we will also use $\mathcal{F}(b)$ for \widehat{b} and $\mathcal{F}(c)$ for \widehat{c} .

Because we do not have a coproduct on the algebras B and C , we do not expect these duals to have a natural algebra structure. However, the extra structure coming from the separability idempotent E , will provide the following properties of these duals.

2.7 Definition We can define a bilinear form on $\widehat{B} \times \widehat{C}$ by

$$\langle \widehat{b}, \widehat{c} \rangle = \langle E, \widehat{b} \otimes \widehat{c} \rangle = (\psi \otimes \varphi)((b \otimes 1)E(1 \otimes c))$$

for $b \in B$ and $c \in C$. \square

A simple calculation gives

$$\langle \widehat{b}, \widehat{c} \rangle = \varphi(S'^{-1}(b)c) = \psi(bS^{-1}(c))$$

for all b, c . This can also be written as

$$\langle \widehat{b}, \widehat{c} \rangle = \langle S'^{-1}(b), \widehat{c} \rangle = \langle \widehat{b}, S^{-1}(c) \rangle$$

where in the right hand side, we use the pairing of C with \widehat{C} and of B with \widehat{B} , coming from evaluation. The last equation means that $\iota \otimes \mathcal{F}$ and $\mathcal{F} \otimes \iota$ transform the pairings of C with \widehat{C} and of B with \widehat{B} respectively to the new pairing of the dual spaces.

We also have the following result about the adjoints of the anti-isomorphisms S and S' .

2.8 Proposition We have linear maps

$$\widehat{S} : \widehat{C} \rightarrow \widehat{B} \quad \text{and} \quad \widehat{S}' : \widehat{B} \rightarrow \widehat{C}$$

given by

$$\widehat{S}(\omega)(b) = \omega(S(b)) \quad \text{and} \quad \widehat{S}'(\omega)(c) = \omega(S'(c))$$

where $b \in B$, $c \in C$ and ω in C and B respectively. We have

$$\widehat{S}(\widehat{b}) = S^{-1}(b) \quad \text{and} \quad \widehat{S}'(\widehat{c}) = S'^{-1}(c).$$

Proof: Take $b, b' \in B$. Then

$$\langle \widehat{S}(\widehat{b}), b' \rangle = \langle \widehat{b}, S(b') \rangle = \psi(bS(b')) = \varphi(b'S^{-1}(b)) = \langle S^{-1}(b), \widehat{b'} \rangle.$$

Similarly for the other formula. \square

In the next section, we will build further on these formulas and relate them with the involutive structure.

3. The involutive case and duality

In this section, we assume that the algebras B and C are $*$ -algebras and that E is a self-adjoint idempotent in $M(B \otimes C)$. Later in this section, we will consider the case where moreover these algebras are *operator algebras*, but for the moment, we consider the more general case and see what can be obtained already then.

We start with the following extra (and expected) relations of the anti-isomorphisms S and S' with the involutive structures.

3.1 Proposition For all $b \in B$ and $c \in C$ we have

$$S'(S(b)^*)^* = b \quad \text{and} \quad S(S'(c)^*)^* = c.$$

Proof: Given $b \in B$ we have $E(b \otimes 1) = E(1 \otimes S(b))$ and if we take adjoints, we find $(b^* \otimes 1)E = (1 \otimes S(b)^*)E$. This implies that $S'(S(b)^*) = b^*$ proving the first formula. Similarly the second one is proven. \square

For the properties of the integrals, we get the following extras.

3.2 Proposition The integrals φ and ψ are self-adjoint.

Proof: By taking the adjoint of $(\iota \otimes \varphi)E = 1$ we find $(\iota \otimes \overline{\varphi})E = 1$ where $\overline{\varphi}$ is defined by $\overline{\varphi}(c) = \varphi(c^*)^-$ for $c \in C$ and where $\overline{\lambda}$ is the complex conjugate of the complex number λ . Now, by the uniqueness of the left integral, we get $\overline{\varphi} = \varphi$ and this proves that φ is self-adjoint. Similarly we have $\overline{\psi} = \psi$ and ψ is self-adjoint. \square

It is a consequence of the uniqueness of the modular automorphisms that then

$$\sigma(c^*) = \sigma^{-1}(c)^* \quad \text{and} \quad \sigma'(b^*) = \sigma'^{-1}(b)^*$$

for all $c \in C$ and $b \in B$. This result also follows from the the formulas in Proposition 3.1 as we have seen in Proposition 2.3 that $\sigma = SS'$ and $\sigma' = (S'S)^{-1}$.

Next, we consider the question of *positivity of the integrals*. Recall that a linear functional ω on a $*$ -algebra A is called positive if $\omega(a^*a) \geq 0$ for all $a \in A$.

First we have the following.

3.3 Proposition If φ is positive on C then ψ is positive on B and vice versa.

Proof: Assume that φ is positive on C . Then for all $b \in B$ we have

$$\begin{aligned}\psi(b^*b) &= (\psi \otimes \varphi)((b^* \otimes 1)E(b \otimes 1)) \\ &= (\psi \otimes \varphi)((1 \otimes S'^{-1}(b^*))E(1 \otimes S(b))) \\ &= \varphi(S'^{-1}(b^*)S(b)) = \varphi(S(b)^*S(b)) \geq 0.\end{aligned}$$

For the last equality, we have used one of the formulas in Proposition 3.1.

Similarly for the other direction. □

This result can also be obtained as follows. Given $b \in B$ we have

$$\begin{aligned}\psi(b^*b) &= \varphi(S(b^*b)) = \varphi(S(b)S(b^*)) \\ &= \varphi(\sigma^{-1}(S(b^*))S(b)) = \varphi(S'^{-1}(b^*)S(b)) \\ &= \varphi(S(b)^*S(b)) \geq 0.\end{aligned}$$

Remark that we have a similar result for $*$ -algebraic quantum groups, but there it is much harder to prove it. The reason is that here, there is a trivial 'scaling constant' essentially because we have

$$\varphi \circ SS' = \varphi \quad \text{and} \quad \psi \circ S'S = \psi.$$

For this result about $*$ -algebraic quantum groups, see e.g. [K-VD] for the original proof and [DC-VD] for a newer simpler one.

Now we want to show that the $*$ -algebras are *operator algebras* if the integrals are positive. Later we will see that the converse is also true (see Section 4). Compare this with a similar property for $*$ -algebraic quantum groups (again see [K-VD]).

We will need a few lemmas before we can show this. First we have the following general (and well-known) result.

3.4 Lemma If ψ is positive on B and φ is positive on C then the linear functional $\psi \otimes \varphi$ is positive on $B \otimes C$.

Proof: Take any $x \in B \otimes C$ and write it as $x = \sum p_i \otimes q_i$ with (p_i) and (q_i) a finite number of elements in B and C respectively. Because φ is positive on C , the matrix with matrix elements $\varphi(q_i^*q_j)$ is positive and therefore it can be diagonalized with a unitary matrix. This allows us to find elements (q'_i) in C and scalars $(c_{k\ell})$ so that

$$\varphi(q_i^*q_j) = \sum_k \overline{c_{ik}}c_{jk}\varphi(q_k'^*q_k').$$

Then we have

$$(\psi \otimes \varphi)(x^*x) = \sum_{i,j} \psi(p_i^*p_j) \varphi(q_i^*q_j) \geq 0$$

by the positivity of ψ on B . □

Next we have the following.

3.5 Lemma Assume again that the integrals are positive. Then for $c, c_1 \in C$ we have

$$\varphi(c^*c_1^*c_1c) \leq \varphi(c_1c_1^*)\varphi(c^*c).$$

Similarly, for $b, b_1 \in B$ we have

$$\psi(b^*b_1^*b_1b) \leq \psi(b_1b_1^*)\psi(b^*b).$$

Proof: We have

$$\begin{aligned} \varphi(c^*c_1^*c_1c) &= (\psi \otimes \varphi)((1 \otimes c^*c_1^*)E(1 \otimes c_1c)) \\ &= (\psi \otimes \varphi)((S'(c_1^*) \otimes c^*)E(S^{-1}(c_1) \otimes c)) \\ &= (\psi \otimes \varphi)((S^{-1}(c_1)^* \otimes c^*)E(S^{-1}(c_1) \otimes c)). \end{aligned}$$

We know that $1 - E = (1 - E)^*(1 - E)$ because E is assumed to be a self-adjoint idempotent. This means that $E \leq 1$ and as $\psi \otimes \varphi$ is positive on $B \otimes C$, the last expression above is dominated by

$$(\psi \otimes \varphi)((S^{-1}(c_1)^* \otimes c^*)(S^{-1}(c_1) \otimes c)).$$

Therefore we get

$$\varphi(c^*c_1^*c_1c) \leq \psi(S^{-1}(c_1c_1^*))\varphi(c^*c)$$

and because $\varphi \circ S = \psi$, we get the result. Similarly for the second formula of the lemma. □

As a consequence we obtain the announced property.

3.6 Proposition If the integrals are positive, the algebras B and C have faithful *-representations on a Hilbert space and are therefore operator algebras.

Proof: Define a scalar product on C by

$$\langle c', c \rangle = \varphi(c^*c').$$

This is a non-degenerate sesquilinear form on C because ψ is positive and faithful. Let \mathcal{H} be the Hilbert space completion of C for this scalar product. Denote the imbedding

of C in \mathcal{H} as $c \mapsto \Lambda(c)$. Define $\pi(c)\Lambda(c') = \Lambda(cc')$ for all c, c' . From Lemma 3.5 we see that

$$\|\pi(c)\Lambda(c')\|^2 \leq \varphi(cc^*)\|\Lambda(c')\|^2$$

for all c, c' . This implies that $\pi(c)$ extends uniquely to a bounded operator on \mathcal{H} . Obviously π will be a faithful $*$ -representation on \mathcal{H} . This proves the result for C . The result for B follows in the same way from the second formula in the previous lemma. \square

We now combine this property with a result we obtained in Section 1 about the structure of the underlying algebras B and C .

3.7 Proposition If the integrals are positive, the algebras B and C are direct sums of matrix algebra.

Proof: We have seen in Section 1 that any element b of B sits in a finite-dimensional two-sided ideal BbB (see a remark following Proposition 1.9). If now b is self-adjoint, then this two-sided ideal will be a $*$ -ideal. As B is an operator algebra, the same is true for these ideals. Then they are direct sums of matrix algebras. It follows that B itself will be a direct sum of matrix algebras. The same argument holds for C . \square

So B has the form $\sum_{\oplus} B_{\alpha}$ where $B_{\alpha} \simeq M_{n(\alpha)}$. Elements in B are functions $\alpha \mapsto b_{\alpha}$ where $b_{\alpha} \in B_{\alpha}$ for all α and with finite support, that is only finitely many elements b_{α} will be non-zero. Clearly C will have the same form (with the same index set and the same dimensions). The choices can be made so that S maps B_{α} to C_{α} and that S' maps C_{α} to B_{α} for all α . The element E will have components E_{α} in each of the tensor products $B_{\alpha} \otimes C_{\alpha}$. In Section 4, where we study examples, we will continue with the more detailed investigation of this case.

There we will also prove the converse result, namely that the integrals will be positive if the underlying algebras are operator algebras. This will complete the characterization of separability idempotents in the case of operator algebras.

Another remark concerns the results on the connection between the existence of traces and the implementation of the modular automorphisms. This becomes obvious here because the algebras are direct sums of matrix algebras. Again for concrete formulas, we refer to Section 4.

Duality

Recall the definitions of the dual spaces (Definition 2.5):

$$\widehat{B} = \{\psi(b \cdot) \mid b \in B\} \quad \text{and} \quad \widehat{C} = \{\varphi(\cdot c) \mid c \in C\}$$

and of the conventions for the Fourier transforms

$$\widehat{b} = \psi(b \cdot) \quad \text{and} \quad \widehat{c} = \varphi(\cdot c)$$

when $b \in B$ and $c \in C$. We also have the adjoint maps $\widehat{S} : \widehat{B} \rightarrow \widehat{C}$ and $\widehat{S}' : \widehat{C} \rightarrow \widehat{B}$. Now, we define the involutions on the dual spaces.

3.8 Definition Define

$$\begin{aligned}\omega^*(b) &= \omega(S(b)^*)^- && \text{when } b \in B \text{ and } \omega \in \overline{C} \\ \omega^*(c) &= \omega(S'(c)^*)^- && \text{when } c \in C \text{ and } \omega \in \overline{B}.\end{aligned}$$

□

Because of the result in Proposition 3.1, we have $\omega^{**} = \omega$ for $\omega \in \widehat{B}$ and for $\omega \in \widehat{C}$. For the Fourier transforms, we find the following formulas.

3.9 Proposition For $b \in B$ and $c \in C$ we get

$$\widehat{b}^* = (S(b^*))^\wedge \quad \text{and} \quad \widehat{c}^* = (S'(c^*))^\wedge.$$

Proof: Take $b \in B$ and $c \in C$. Then

$$\begin{aligned}\langle \widehat{b}^*, c \rangle &= \langle \widehat{b}, S'(c)^* \rangle^- = \psi(bS'(c)^*) \\ &= \psi(S'(c)b^*) = \varphi(S'^{-1}(b^*)c) \\ &= \varphi(cS(b^*)) = \langle (S(b^*))^\wedge, c \rangle.\end{aligned}$$

This proves the first formula of the proposition. The second one is proven in a similar way. □

Next we combine the bilinear form that we defined on $\widehat{B} \times \widehat{C}$ as

$$\langle \widehat{b}, \widehat{c} \rangle = \langle E, \widehat{b} \otimes \widehat{c} \rangle,$$

for $b \in B$ and $c \in C$, with the involutive structure.

3.10 Proposition We can define a sesquilinear form on $\widehat{C} \times \widehat{C}$ by

$$\langle \widehat{c}_1, \widehat{c}_2 \rangle = \langle E, (\widehat{c}_2)^* \otimes \widehat{c}_1 \rangle.$$

It satisfies

$$\langle \widehat{c}_1, \widehat{c}_2 \rangle = \langle c_1, c_2 \rangle$$

for all c_1, c_2 in C where the last form is $\varphi(c_2^*c_1)$ as before. Similarly for B .

Proof: Take $c_1, c_2 \in C$. Then

$$\begin{aligned}\langle E, (\widehat{c}_2)^* \otimes \widehat{c}_1 \rangle &= \langle E, \widehat{S'(c_2^*)} \otimes \widehat{c}_1 \rangle \\ &= (\psi \otimes \varphi)((S'(c_2^*) \otimes 1)E(1 \otimes c_1)) \\ &= (\psi \otimes \varphi)((1 \otimes c_2^*)E(1 \otimes c_1)) \\ &= \varphi(c_2^*c_1).\end{aligned}$$

□

We get Plancherel type formulas for these 'Fourier transforms'.

Another remark concerns the results on the connection between the existence of traces and the implementation of the modular automorphisms as found in Proposition 2.4 of the previous section. This becomes obvious when the algebras are direct sums of matrix algebras. Again, for concrete formulas, we refer to Section 4.

4. Examples and special cases

In this section, we will consider examples to illustrate this theory. We will also treat some special cases, related with these examples.

First we modify the basic example that we treated in the beginning of the introduction to motivate our paper. It will turn out to be a building block for more examples. In fact, as we announced already in the previous section, we will see that any separability idempotent in the involutive case with positive integrals, is obtained using these building blocks.

Examples with matrix algebras

The following is certainly well-known and not very deep. But we need to cover this case as it gives an essential part for some other examples.

4.1 Example Let $n \in \mathbb{N}$ and take for B and C the $*$ -algebra $M_n(\mathbb{C})$ of all $n \times n$ complex matrices. Let r be any invertible element in $M_n(\mathbb{C})$ and assume that $\text{Tr}(r^*r) = n$ where Tr is the trace on $M_n(\mathbb{C})$, normalized so that $\text{Tr}(1) = n$ as before. Now define

$$E = (r \otimes 1)E_0(r^* \otimes 1)$$

where

$$E_0 = \frac{1}{n} \sum_{i,j} e_{ij} \otimes e_{ij}$$

as in the introduction. Again (e_{ij}) are matrix units in $M_n(\mathbb{C})$.

i) We find

$$E^2 = (r \otimes 1)E_0(r^*r \otimes 1)E_0(r^* \otimes 1)$$

and because

$$\begin{aligned} E_0(r^*r \otimes 1)E_0 &= \frac{1}{n^2} \sum_{i,j,\ell,k} e_{ij}r^*re_{\ell k} \otimes e_{ij}e_{\ell k} \\ &= \frac{1}{n^2} \sum_{i,j,k} e_{ij}r^*re_{jk} \otimes e_{ik} \\ &= \frac{1}{n^2} \text{Tr}(r^*r) \sum_{i,k} e_{ik} \otimes e_{ik} = E_0, \end{aligned}$$

we see that actually E is a self-adjoint idempotent in $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$. Observe that in the case $r = 1$, we get the idempotent E as considered in the introduction.

ii) It is now straightforward to verify that E is a separability idempotent in $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$. We calculate the *anti-isomorphisms* S and S' . We use b and c to denote elements in $M_n(\mathbb{C})$ but keeping in mind that $b \in B$ and $c \in C$. We use S_0 for the anti-isomorphism of $M_n(\mathbb{C})$ given by $S_0(e_{ij}) = e_{ji}$ (i.e. transposition of matrices). Then we find

$$\begin{aligned} E(b \otimes 1) &= (r \otimes 1)E_0(r^*b \otimes 1) \\ &= (r \otimes 1)E_0(r^* \otimes S_0(r^*br^{*-1})) \end{aligned}$$

so that $S(b) = S_0(r^*br^{*-1})$ for all b . Similarly

$$\begin{aligned} (1 \otimes c)E &= (r \otimes c)E_0(r^* \otimes 1) \\ &= (rS_0(c)r^{-1}r \otimes 1)E_0(r^* \otimes 1) \end{aligned}$$

and we see that $S'(c) = rS_0(c)r^{-1}$ for all c .

iii) We claim that the integrals are given by

$$\psi = n\text{Tr}(p \cdot) \quad \text{and} \quad \varphi = n\text{Tr}(q \cdot)$$

where $p = (rr^*)^{-1}$ and $q = S_0(r^*r)^{-1}$. We will prove the formula for ψ . The other one is proven in a completely similar way. We have

$$\begin{aligned} (\text{Tr} \otimes \iota)((p \otimes 1)E) &= (\text{Tr} \otimes \iota)((pr \otimes 1)E_0(r^* \otimes 1)) \\ &= (\text{Tr} \otimes \iota)((r^{*-1} \otimes 1)E_0(r^* \otimes 1)) \\ &= (\text{Tr} \otimes \iota)E_0 \\ &= \frac{1}{n} \sum_{ij} \text{Tr}(e_{ij})e_{ij} = \frac{1}{n} \sum_j e_{jj} \end{aligned}$$

and this proves the formula for ψ . Observe that these integrals are positive because the elements p and q are positive matrices and the trace is positive.

iv) For the *modular automorphisms* we get

$$\begin{aligned} \sigma(c) &= SS'(c) = S(rS_0(c)r^{-1}) \\ &= S_0(r^*rS_0(c)r^{-1}r^{*-1}) \\ &= qcq^{-1} \end{aligned}$$

for all c because $q = S_0(r^*r)^{-1}$. We have used that $S_0^2 = \iota$. Similarly we find

$$\begin{aligned} \sigma'^{-1} &= S'S(b) = S'(S_0(r^*br^{*-1})) \\ &= rr^*br^{*-1}r^{-1} = pbp^{-1} \end{aligned}$$

for all b and we see that $\sigma'(b) = pbp^{-1}$ where now $p = (rr^*)^{-1}$.

One can verify that $\varphi(cc') = \varphi(c'\sigma(c))$ with $\sigma(c) = qcq^{-1}$, obtained before and similarly for ψ . Indeed, in this case, the modular automorphisms determine the integrals up to scalar. One also verifies that $\psi = \varphi \circ S$ and $\varphi = \psi \circ S'$. This is an easy consequence of the equations

$$S(p) = q \quad \text{and} \quad S'(q) = p$$

that are easily verified. □

This example is universal in the following sense.

4.2 Proposition Any self-adjoint separability idempotent in $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ is as in the previous example.

Proof: Let F be any self-adjoint separability idempotent in $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$. Let S_F and S'_F be the associated anti-isomorphisms. The automorphism $S'_F S_0$ of $M_n(\mathbb{C})$ has the form $b \mapsto rbr^{-1}$ for some invertible matrix $r \in M_n(\mathbb{C})$. Then $S'_F(c) = rS_0(c)r^{-1}$ for all c . We know from the general theory (see Proposition 3.1 in Section 3) that $S_F(b) = S'_F{}^{-1}(b^*)^*$ and it will follow that

$$S_F(b) = S'_F{}^{-1}(b^*)^* = S_0(r^{-1}b^*r)^* = S_0(r^*br^{*-1})$$

for all b . Because we got the same expressions for the anti-isomorphisms in the example above, it follows from Proposition 1.12 in Section 1, that r can be scaled so that $\text{Tr}(r^*r) = n$ and $F = (r \otimes 1)E_0(r^* \otimes 1)$. □

We need to make the following remark. Consider again the example. Let $r = v|r|$ be the *polar decomposition* of r . So $|r| = (r^*r)^{\frac{1}{2}}$, it is a positive self-adjoint invertible matrix and v is a unitary matrix. Then we have $E = (\alpha_v \otimes \iota)F$ where $F = (|r| \otimes \iota)E_0(|r| \otimes 1)$ and where α_v is the $*$ -automorphism $b \mapsto vbv^*$. It follows from this observation that we can assume in the previous proposition, up to a $*$ -isomorphism, that r is a positive self-adjoint invertible matrix. This simplifies some formulas. We then have e.g. that $q = S_0(p)$.

4.3 Example Let $n \in \mathbb{N}$ and take again for B and C the algebra $M_n(\mathbb{C})$ as in Example 4.1. We do not consider the involution now. Let r and s be any two invertible matrices in $M_n(\mathbb{C})$ and put

$$E = (r \otimes 1)E_0(s \otimes 1)$$

where E_0 is again as in Example 4.1. The same calculation as before gives that $E^2 = \frac{1}{n}\text{Tr}(sr)E$. We see that either $E^2 = 0$ or that we can scale the matrices r and s in such a way that $\text{Tr}(sr) = n$ so that $E^2 = E$.

In the last case, we have a separability idempotent. A straightforward calculation gives again the formulas for the anti-isomorphisms and the modular automorphisms. We find

$$S(b) = S_0(sbs^{-1}) \quad \text{and} \quad S'(c) = rS_0(c)r^{-1}$$

and

$$\sigma'(b) = p b p^{-1} \quad \text{and} \quad \sigma(c) = q c q^{-1}$$

where $p = (rs)^{-1}$ and $q = S_0(sr)^{-1}$. For the left and right integrals, we get the same formulas as in Example 4.1:

$$\psi = n \text{Tr}(p \cdot) \quad \text{and} \quad \varphi = n \text{Tr}(q \cdot).$$

where p and q are as defined above. □

Remark that in the case where $\text{Tr}(sr) = 0$, so that $E^2 = 0$, we still have the anti-isomorphisms S and S' , as well as the left and the right integrals and the associated modular automorphisms. This suggests that there might be a more general and still interesting theory where full elements E in $M(B \otimes C)$ are considered, without the assumption of having an idempotent element. However, as we see from the first example and Proposition 4.2, this situation will not be possible in the $*$ -algebra case. See Section 5 where we discuss possible future research.

Also here, we have that Example 4.3 gives every separability idempotent in the case where B and C are $M_n(\mathbb{C})$. This is the content of the following proposition.

4.4 Proposition Any separability idempotent in $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ is as in the previous example.

Proof: Let F be any separability idempotent in $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$. There exist invertible matrices s and r such that

$$S'_F(c) = r S_0(c) r^{-1} \quad \text{and} \quad S_F(b) = S_0(s b s^{-1})$$

for all b, c . It follows from Proposition 1.12 that r and s can be scaled so that $\text{Tr}(sr) = 1$ and that $F = (r \otimes 1) E_0 (s \otimes 1)$. □

Again, if we allow automorphisms of the factors, we can even assume in this case that either r or s is equal to 1. The argument is as given for the remark following Proposition 4.2.

Examples with direct sums

We will now use the above examples to build new ones. First we consider a general construction.

Let J be an index set. Assume that for all $\alpha \in J$ we have a pair of non-degenerate algebras B_α and C_α . We consider the direct sums

$$B = \sum_{\alpha \in J} B_\alpha \quad \text{and} \quad C = \sum_{\alpha \in J} C_\alpha.$$

Elements b in B are of the form $(b_\alpha)_{\alpha \in J}$ where $b_\alpha \in B_\alpha$ for all $\alpha \in J$ and where only a finite number of the components is non-zero. Similarly for C . There are natural identifications

$$M(B) = \prod_{\alpha \in J} M(B_\alpha) \quad \text{and} \quad M(C) = \prod_{\alpha \in J} M(C_\alpha).$$

Elements in $M(B)$ have the form $(x_\alpha)_{\alpha \in J}$ where $x_\alpha \in M(B_\alpha)$ for all $\alpha \in J$ (without further restrictions) and similarly for elements in $M(C)$. For the tensor product $B \otimes C$ in its multiplier algebra $M(B \otimes C)$ we get

$$B \otimes C = \sum_{\alpha, \beta} B_\alpha \otimes C_\beta$$

$$M(B \otimes C) = \prod_{\alpha, \beta} M(B_\alpha \otimes C_\beta).$$

The basic result using these building blocks is now the following. We use the above notations.

4.5 Proposition For all $\alpha \in J$ let E_α be a separability idempotent in $M(B_\alpha \otimes C_\alpha)$. Define the element E in $M(B \otimes C)$ with components E_α . Then E is a separability idempotent in $M(B \otimes C)$. The anti-isomorphisms S and S' are given by $S(b)_\alpha = S_\alpha(b_\alpha)$ and $S'(c)_\alpha = S'_\alpha(c_\alpha)$ for all $b \in B$ and $c \in C$ (where of course S_α and S'_α are the anti-isomorphisms associated with the components E_α). The left and right integrals on C and B are given by

$$\psi(b) = \sum_{\alpha \in J} \psi_\alpha(b_\alpha) \quad \text{and} \quad \varphi(c) = \sum_{\alpha \in J} \varphi_\alpha(c_\alpha).$$

Observe that these sums are finite sums as only finitely many terms are non-zero. Finally, the modular automorphisms σ and σ' are given by $\sigma(b)_\alpha = \sigma_\alpha(b_\alpha)$ and $\sigma'(c)_\alpha = \sigma'_\alpha(c_\alpha)$ for all $b \in B$ and $c \in C$ (where σ_α and σ'_α are the modular automorphisms associated with the components E_α). \square

The proof is straightforward and we leave it mostly as an exercise for the reader. We just make a few remarks.

Clearly E is an idempotent as all its components are idempotents. If $c \in C$, we find that $(E(1 \otimes c))_{\alpha, \beta} = 0$ if $\alpha \neq \beta$ while $(E(1 \otimes c))_{\alpha, \alpha} = E_\alpha(1 \otimes c_\alpha)$ and this will be 0 for all but finitely many α . It follows that $E(1 \otimes c) \in B \otimes C$. Similarly for the three other cases. To prove that E is full, take e.g. a fixed index α_0 and assume that c is the element with 0 in all components, except for the α_0 component where it is an element c_0 of C_{α_0} . For any functional ω on C we will find that $(\iota \otimes \omega)(E(1 \otimes c))$ will only have one non-zero component in the index α_0 , given by $(\iota \otimes \omega_{\alpha_0})(E_{\alpha_0}(1 \otimes c_0))$ where ω_{α_0} is the restriction of ω to the component C_{α_0} . As we can reach all elements in B_{α_0} in such a way, we see that the left leg of E is indeed all of B . Similarly for the right leg.

It is obvious that $E(b \otimes 1) = E(1 \otimes S(b))$ if we define S on B by $S(b)_\alpha = S_\alpha(b_\alpha)$ for all α . Similarly for S' on C . This will show that we have a separability idempotent. The formulas for the modular automorphisms are obvious. And the same is true for the formulas for the integrals.

It is also clear that this construction will work in the involutive case as well. Indeed, if components are $*$ -algebras, then so will be B and C and if each E_α is self-adjoint, the same will be true for E .

This result can now be combined with the examples given in 4.1 and 4.3. to obtain infinite-dimensional cases. Indeed, we can apply Proposition 4.5 with the examples in Example 4.1 (in the involutive case) or with Example 4.3 (in the general case).

In the involutive case, the integrals are positive. And now we show that conversely, every self-adjoint separability idempotent with positive integrals, has to be of that form.

4.6 Theorem Assume that B and C are $*$ -algebras and that E is a self-adjoint separability idempotent in $M(B \otimes C)$ with positive integrals. Then B and C are direct sums of matrix algebras and E is the direct sum of building blocks as in Proposition 4.5 with each component like in Example 4.1.

Proof: We have seen in Proposition 3.7 that B and C are a direct sum matrix algebras. Because they are anti-isomorphic, we can assume e.g. that there is an index set J so that

$$B = \sum_{\alpha \in J} B_\alpha \quad \text{and} \quad C = \sum_{\alpha \in J} C_\alpha$$

where for each $\alpha \in J$ we have a natural number $n(\alpha)$ and B_α and C_α isomorphic with $M_{n(\alpha)}$. We can also assume that $S : B_\alpha \rightarrow C_\alpha$ and $S' : C_\alpha \rightarrow B_\alpha$ for each α . Then E will be the direct sum of its components $E_{\alpha,\alpha}$. We know by Proposition 4.2 that this component has to be of the form as in Example 4.1. This completes the proof. \square

Discrete quantum groups

For our next example, we have the following. We refer to [VD1] and [VD2] for the theory of multiplier Hopf algebras and in particular to [VD-Z] for multiplier Hopf algebras of discrete type.

4.7 Proposition Let (A, Δ) be a regular multiplier Hopf algebra of discrete type. Assume that the (left) cointegral h is an idempotent. Then $\Delta(h)$ is a separability idempotent in $M(A \otimes A)$. The anti-isomorphisms S and S' coincide with the antipode of A . The integrals φ and ψ are the left and right integrals on A respectively.

Proof: Recall that by definition, a left cointegral h exists. However, it can happen that $h^2 = 0$ (when $\varepsilon(h) = 0$ where ε is the counit). If this is not the case, we can scale h so that $\varepsilon(h) = 1$. Then $h^2 = h$ and h will also be a right cointegral. This is what we assume here. We will use the term discrete quantum group for such multiplier Hopf algebras of discrete type with normalized cointegral.

It follows of course that $\Delta(h)$ is an idempotent in $M(A \otimes A)$. From the theory of discrete quantum groups, we know that

$$\Delta(h)(a \otimes 1) = \Delta(h)(1 \otimes S(a)) \quad \text{and} \quad (1 \otimes a)\Delta(h) = (S(a) \otimes 1)\Delta(h)$$

for all a where S is the antipode of A . The first equality is true for any left cointegral and the second for any right cointegral. In this case, these are the same.

Because we assume that (A, Δ) is regular, we will have that $\Delta(h)$ is a regular idempotent. And it is full because, again from the theory of discrete quantum groups, we know that the legs of $\Delta(h)$ are all of A . Therefore we do have a regular separability idempotent.

It is clear from the formulas above that the antipode S of A gives the anti-isomorphisms S and S' . And it is also clear the left and right integrals on A give the integrals for the separability idempotent $\Delta(h)$. \square

If (A, Δ) is a multiplier Hopf $*$ -algebra of discrete type, it is automatically regular, left and right cointegrals coincide and are self-adjoint idempotents. In this case, we get a self-adjoint separability idempotent. We know that Theorem 4.6 applies.

Again remark that this case also suggest that a theory where it is not assumed that E is idempotent, but where it is allowed that $E^2 = 0$ might be possible.

Weak multiplier Hopf algebras

Now we consider the case of weak multiplier Hopf algebras. We refer to [VD-W2], [VD-W3] and [VD-W4] for the theory of weak multiplier Hopf algebras. We have the following result.

4.8 Proposition Let (A, Δ) be a regular weak multiplier Hopf algebra. Denote by B the image $\varepsilon_s(A)$ of A under the source map ε_s and by C the image $\varepsilon_t(A)$ of A under the target map ε_t . Then the canonical idempotent E is a separability idempotent in $M(B \otimes C)$. The antipode S of A , when restricted to B and C gives the associated anti-isomorphisms S and S' respectively. The left and right integrals on C and B are given by the maps

$$\varepsilon_t(a) \mapsto \varepsilon(a) \quad \text{and} \quad \varepsilon_s(a) \mapsto \varepsilon(a)$$

here ε is the counit of A . The modular automorphisms are the restrictions of the square S^2 of the antipode.

Proof: We will first show that E is a separability idempotent.

i) We know that E is an idempotent element in the multiplier algebra $M(A \otimes A)$ by definition. However, it has been shown in Proposition 2.13 of [VD-W4] that actually $E \in M(B \otimes C)$ where $B = \varepsilon_s(A)$ and $C = \varepsilon_t(A)$.

ii) In the same proposition of [VD-W4], it is also proven that all four elements of the form $E(y \otimes 1)$, $(y \otimes 1)E$, $(1 \otimes x)E$, $E(1 \otimes x)$ are in $B \otimes C$ for all $y \in B$ and $x \in C$. This proves that E satisfies Assumption 1.1 and that it is regular.

iii) It is already shown in Lemma 3.2 of [VD-W3] that B and C are the left and the right leg of E .

iv) Finally, again in Proposition 2.13 of [VD-W4] we find that

$$E(y \otimes 1) = E(1 \otimes S(y)) \quad \text{and} \quad (1 \otimes x)E = (S(x) \otimes 1)E$$

for $y \in B$ and $x \in C$ where S is the antipode of A , extended to the multiplier algebra $M(A)$.

This proves that E is a separability idempotent.

vi) In the proof of Proposition 3.2 in [VD-W4], it is shown that there are linear functionals f on B and g on C given by

$$f(\varepsilon_s(a)) = \varepsilon(a) \quad \text{and} \quad g(\varepsilon_t(a)) = \varepsilon(a)$$

for all $a \in A$. They were shown to satisfy the invariance properties

$$(f \otimes \iota)E = 1 \quad \text{and} \quad (\iota \otimes g)E = 1.$$

Hence, they are respectively the right and the left integrals.

vii) By the general theory, we know that the modular automorphisms are given by S^2 . \square

Remark that these integrals on B and C also exist when there are no integrals on A . This is different from the situation we had in the previous example with a discrete quantum group. In that case, the integrals on the legs of E where the integrals on the original algebra A .

Also remark that this example is in fact essentially the general situation. Indeed, if we have a separability idempotent $E \in M(B \otimes C)$, we can make the algebra $C \otimes B$ into a regular weak multiplier Hopf algebra so that the canonical idempotent is $1 \otimes E \otimes 1$ where the first 1 is the identity in $M(C)$ and the second 1 the identity in $M(B)$. For this example, one has precisely that $B \otimes 1$ and $1 \otimes C$ are the images of the source and target maps respectively. See Proposition 3.3 in [VD-W4].

5. Conclusions and further research

In this paper, we have studied separability idempotents E in the multiplier algebra $M(B \otimes C)$ where B and C are non-degenerate algebras over the complex numbers. One of the ingredients of the theory are the anti-isomorphisms $S : B \rightarrow C$ and $S' : C \rightarrow B$ given by the formulas

$$E(b \otimes 1) = E(1 \otimes S(b)) \quad \text{and} \quad (1 \otimes c)E = (S'(c) \otimes 1)E$$

for $b \in B$ and $c \in C$. Another important feature is the existence of unique and faithful left and right integrals φ and ψ on C and B respectively, satisfying

$$(\iota \otimes \varphi)E = 1 \quad \text{and} \quad (\psi \otimes \iota)E = 1$$

in $M(B)$ and $M(C)$ respectively.

A first example comes from a discrete quantum group (A, Δ) and $E = \Delta(h)$ where h is the normalized cointegral. A second example comes from a regular weak multiplier Hopf algebra (A, Δ) . Now E is the canonical element and B and C are the images of the source and target maps. In fact, the second example is more or less universal in the sense that essentially every separability idempotent is of this form.

In the involutive case and when the integrals are positive, the situation is completely understood. This happens when B and C are direct sums of matrix algebras with the natural involutive structure. When the integrals are not positive, it probably makes little sense to consider the involutive structure.

In the general case, there seems to be a more general concept, suggested at various places in the paper. Look at the following example.

5.1 Example Consider the case of a regular multiplier Hopf algebra (A, Δ) of discrete type with a right cointegral k satisfying $k^2 = 0$. This happens when $\varepsilon(k) = 0$. Again let $E = \Delta(k)$. Of course, E will no longer be an idempotent, but on the other hand, it will share many properties of a separability idempotent.

Indeed, first of all we have that

$$\begin{aligned} \Delta(k)(a \otimes 1) &= \sum_{(a)} \Delta(ka_{(1)})(1 \otimes S(a_{(2)})) \\ &= \sum_{(a)} \Delta(k)(1 \otimes \varepsilon(a_{(1)})S(a_{(2)})) \\ &= \Delta(k)(1 \otimes S(a)) \end{aligned}$$

for all a in A . On the other hand, by the uniqueness of the right cointegral, there exists a homomorphism $\gamma : A \rightarrow \mathbb{C}$ satisfying $ak = \gamma(a)k$ for all a . If the right integral is also a left integral, this would mean that $\gamma = \varepsilon$ but in general this need not be the case. Then we find

$$\begin{aligned} (1 \otimes a)\Delta(k) &= \sum_{(a)} (S(a_{(1)}) \otimes 1)\Delta(a_{(2)}k) \\ &= \sum_{(a)} (\gamma(a_{(2)})S(a_{(1)}) \otimes 1)\Delta(k) \\ &= (S'(a) \otimes 1)\Delta(k) \end{aligned}$$

if we define $S'(a) = \sum_{(a)} \gamma(a_{(2)})S(a_{(1)})$ for all a . It turns out that we have anti-isomorphisms $S : A \rightarrow A$ and $S' : A \rightarrow A$ satisfying

$$E(a \otimes 1) = E(1 \otimes S(a)) \quad \text{and} \quad (1 \otimes a)E = (S'(a) \otimes 1)E$$

for all a . Also the Assumption 1.1 in Section 1 is fulfilled for this element E (because we have a regular coproduct Δ on A). The element will also be full and it will satisfy the conditions in Definition 1.5, except that it will not be an idempotent. \square

And here is another example of a similar situation.

5.2 Example As in Example 4.3, take $E = (r \otimes 1)E_0(s \otimes 1)$. We still assume r and s to be invertible, but now we take the case where $\text{tr}(sr) = 0$. As mentioned already, this will give $E^2 = 0$ but most of the other properties of E remain true as in Example 5.1. \square

If a theory is possible including the above cases, one may wonder whether a similar generalization would also be possible for weak multiplier Hopf algebras. At first sight, it does not seem very likely because E stands for $\Delta(1)$ in that case and of course this is always an idempotent.

Another possible direction for further research is of course the case where non-degenerate algebras over more general fields are considered. In this case more work need to be done to relate the theory as developed in this paper with existing literature.

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